

FRONTIER PARETO OPTIMAL POINTS OF DISCRETE MULTIOBJECTIVE OPTIMIZATIONS

Asist. prof. dr. Angelova J.¹, Prof. dr. eng. Malakov I.²,
Dept. of Mathematics, UCTM - Sofia¹,
Dept. ADP, TU - Sofia, Bulgaria²

Abstract: This paper considers and systemizes widely used criteria for multiobjective optimization applied to solving discrete minimization (maximization) problems. These problems arise in science, technique, economy and others when choosing the optimal design variant of a complex system. The effectiveness of designed system significantly depends on the chosen solutions of the problems. The usage of optimality principles for determining Pareto effective solutions is shown on a bi-objective discrete minimization problem. Some procedures of cluster analysis are proposed to support processing of final solution of a decision maker.

KEYWORDS: DISCRETE MULTIOBJECTIVE OPTIMIZATION, PARETO SET, OPTIMALITY PRINCIPLES, CLUSTER ANALYSIS

1. Introduction

Choice of an optimal (effective, rational) solution from a set of possible ones is a typical problem in different fields of technique, economy, transport, management, ecology, military and others during design of a complex system [5, 7, 8, 12], e.g. maximum utility, diversity, protection and etc for minimum cost, structural performance design in engineering and others. Real conditions require the choice to be accomplished by a variety of criteria, as the investigated objects are evaluated by different contradictory parameters, i.e. the improvement of a design of one parameter leads to deteriorations of others and vice versa.

Therefore the choice of optimal constructive variant is a multiobjective optimization problem (MOP) of a discrete programming. Its practical importance provokes creation of significant number of methods and algorithms, e.g. [1, 7, 8, 9]. The actuality of effective solving MOP leads to many new research works permanently published in specialized literature.

There are many algorithmic and computational problems in the multiobjective optimization. Main of them is the optimality criterion (principle) choice [5, 7]. The optimality criteria define characteristics of preferable solutions and answers to the question of advantage of one solution over another. The choice of optimality criteria depends on the terms of given tasks, user and decision maker (DM) preferences and priorities [4, 7]. In many cases these principles are consecutively applied and/or combined so that they satisfy users and DMs requirements. There are many procedures of determining criteria weights with relative and absolute normalization of scores. The more solutions of MOP are obtained the bigger is the probability of finding the satisfactory solution.

The target of this report is to systemize methodologically the classical optimality criteria for multi-objective decision making (multicriteria decision analysis) and propose some statistic classification procedures leading DMs and users in their final choice.

2. Statement of the problem

The object under investigation is a MOP with m linear objective functions (criteria) in n variables defined in discrete number sets with different cardinality,

minimize vector function $f(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$

subject to $x^{(i)} \in X^{(i)}$,

where:

$f, f: R^n \rightarrow R^m$, is a vector optimality criterion (cost function);

$f_i, f_i: R^n \rightarrow R$, is i -th objective function that has to be minimized, $i=1,2,\dots,m$;

$x_j^{(i)}, x_j^{(i)} \in X_j^{(i)} = \{x_{j1}^{(i)}, x_{j2}^{(i)}, \dots, x_{jl_j}^{(i)}\} \subset [0,1]$, is j -th

subjective variables of function $f_i, i=1,2,\dots,m$;

l_j is the number of values of variable $x_j^{(i)}$ of function f_i .

Symbolically this MOP is stated as

$$\begin{cases} f = (f_1, f_2, \dots, f_m) \rightarrow \min \\ f_i(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) = \sum_{j=1}^n x_j^{(i)} \\ x^{(i)} \in X^{(i)}, \quad i=1,2,\dots,m \end{cases} \quad (1)$$

Values $j^{(i)}$ ($i=1,2,\dots,m$) of subjective variables are stated as m tables of size $n \times l$, where l is the largest cardinality of the definition domains of variables, $l = \max_j l_j$. From these values we

obtain scaled (normed) data $x_{jl}^{(i)} \in [0,1]$ ($j=1,2,\dots,n, l=1,2,\dots,l_j$) that are dimensionless and reduced to uniform measurable unit. For this purpose some normalizing methods are applied, see [7, 12].

A solution of problem (1) belongs to a set of compromise solutions (set of non-dominated, non-improvement, π -optimal alternatives in criterion space and Pareto efficient or optimal in decision space) according to Pareto optimality criterion. The solutions of this set simultaneously cannot be improved by all criteria, i.e. altering efficient solution no objective value can be improved without deteriorating other objective value/values [10]. Express differently there is no other admissible solution with vector objective components better than compromise solutions [2, 9].

The Pareto set $P(X)$ in subjective (decision) space for minimization is defined as follows:

$P(X) = \{x \in X : \text{does not exist } x' \in X \text{ such that } f(x') \leq f(x)\}$,

where X is a solution set [2, 9]. If we consider a maximization problem the inequality sign is in opposite direction. The corresponding set in objective (criterion) space $Y=f(X)$ is introduced by statement:

$P(Y) = \{y \in Y : \text{does not exist } y' \in Y \text{ such that } y' \leq y\}$,

that mean $y' \leq y \Leftrightarrow y'_i \leq y_i, i=1,2,\dots,m$ and $y \neq y'$,

i.e. there exists $i_0 \in \{1,2,\dots,m\}$ such that $y'_{i_0} < y_{i_0}$.

In maximization problems all inequalities are opposite.

On fig.1 by solid circles are presented a set of compromise solutions in objective space of $f_1(x)$ and $f_2(x)$ for a bi-objective maximization problem

$(f_1(x), f_2(x)) \rightarrow \max$,

where criterion functions are normalized.

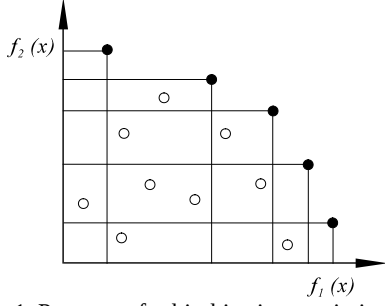


Fig. 1. Pareto set for bi-objective maximization.

A unique solution that optimizes each objective functions of MOP is a rare exception, and that is why it is necessary to introduce a binary relation „preference” ($\succ$) to rank compromise solutions of MOP. For this purpose a different optimality principles (criteria) are applied. The additional information supporting DM in his final choice is needed.

Generally multiobjective decision problems can be stated as

order solution set X of a MOP by preference relation $\succ$ in accordance with preference relation $\succ_Y$ on $Y = f(X)$ or symbolically (see [9])

$$\left| \begin{array}{l} (X, f, \succ) \\ (Y, \succ_Y) \end{array} \right|, \quad (2)$$

where:

X is a solution set;

$f, f: R^n \rightarrow R^m$, is an objective vector function;

$\succ$ is a binary relation of strictly dominates of DM on X ,

i.e. $x_1 \succ x_2$ means that DM strictly prefers solution x_1 to x_2 ;

$Y = f(X)$ is a range of vector function of solutions X ;

$\succ_Y$ is a binary relation of strictly preference of DM on Y ,

i.e. $x_1 \succ x_2 \Leftrightarrow y_1 \succ_Y y_2; y_1 = f(x_1), y_2 = f(x_2)$.

3. Reducing of MOPs to single optimization problem

To choose a satisfactory solution of a MOOP a DM has to:

- obtain solutions of problem (1) that are derived by some rules (optimality principles, algorithms and etc.);
- define problem (2);
- rank $P(Y)$ and
- propose to users the „best” solution or a set of recommendable solutions $C(Y) \subset P(Y)$, as the final choice from $C(Y)$ the user makes by himself.

In practice different methods and algorithms are used to obtain solutions of MOP such as: lexicographical optimization, methods with utility and regret functions (constructing a single aggregate objective function), minimax procedures and others. Usually a MOP is reduced to a single-objective optimization problem (SOP) (scalarization a vector optimization problem to single scalar problem), see e.g. [1, 6] as follows:

minimize scalarized function of weighted criteria F
subject to $x \in D$,

where:

$F, F: R^m \rightarrow R$ is an aggregate objective function (AOF);

$w \in R^m$, $w_1 + w_2 + \dots + w_m = 1$, $w_i \in [0, 1]$, is a weight vector defined significance (importance) of criteria;

$f, f: R^n \rightarrow R^m$, is an objective vector function;

$x = (x_1, x_2, \dots, x_n)$ is a variable vector;

D is a set of admissible solutions of MOP. Obviously the solution set $X \subseteq D$.

Symbolically this problem is written as

$$\left| \begin{array}{l} F(w, f(x)) \rightarrow \min \\ x \in D \end{array} \right|. \quad (3)$$

As scalarized functions (AOFs) are used:

$$F = \sum_{i=1}^m w_i f_i \quad - \text{weighted additive AOF (weighted sum method);}$$

$$F = \prod_{i=1}^m f_i^{w_i} \quad - \text{weighted multiplicative AOF;}$$

$$F = \left(\frac{1}{m} \sum_{i=1}^m |w_i f_i|^p \right)^{1/p}, \quad p > 0, \quad - \text{weighted root-power mean AOF. More AOFs considered as quasi-arithmetic mean aggregation operator one can find in [6].}$$

Other applied methods are methods of ideal, anti-ideal point and minimax procedures.

Minimization problems can be reduced to SOOP as:

$$\arg \min_{x \in D} F(w, f(x) - f^*) \quad \text{or}$$

$$\arg \max_{x \in D} F(w, f(x) - f^*),$$

where:

$\arg \min$ stands for arguments of minimal function's value;

F is a one of mentioned above AOF;

$f^* = (f_1^*, f_2^*, \dots, f_m^*)$ is an objective vector with minimal values (ideal point for minimization), $f_i^* = \min_{x \in D} f_i(x)$;

$\arg \max$ stands for arguments of maximal function's value;

$f^* = (f_1^*, f_2^*, \dots, f_m^*)$ is an objective vector with maximal values (anti-ideal point for minimization), $f_i^* = \max_{x \in D} f_i(x)$.

A minimax problem can be formulated as

$$\arg \min_{x \in D} \{ \max_i F(w, f(x)) \},$$

where maximization is performed in objective space and F is usually of additive type.

To rank preferable solutions $C(X), C(X) \subseteq P(X) \subseteq X$, see problem (2), an aggregate preference function can be used, thus, considering minimization, methods are formulated in the following way [4].

The principle of justified compromise (II1) states, that solution x_1 is preferred to x_2 , $x_1 \succ x_2$, if $\|f(x_1)\|_1 < \|f(x_2)\|_1$, or $\|wf(x_1)\|_1 < \|wf(x_2)\|_1$, if we have weight vector w . In general

$$x_1 \succ x_2, \text{ if } \|wf(x_1)\|_p < \|wf(x_2)\|_p,$$

where $\|\cdot\|_p$ is a p -norm, $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$, $x \in R^n$, $p \geq 1$.

A principle of relative compromise II2 gives, that a solution x_1 is preferred to x_2 ,

$$x_1 \succ x_2, \text{ if } \prod_{i=1}^m f_i^{w_i}(x_1) < \prod_{i=1}^m f_i^{w_i}(x_2),$$

where w is a weight vector.

A principle of ideal point II3 (as in numerical example ideal points derives by minimal values of parameters by all levels per criteria) set that a solution x_1 is „better” than x_2 ,

$$x_1 \succ x_2, \text{ if } \|wf(x_1) - wf^*\|_p < \|wf(x_2) - wf^*\|_p,$$

where f^* is the ideal point and w is a weight vector.

Analogously a principle of anti-ideal point II4 states, that

$$x_1 \succ x_2, \text{ if } \|wf(x_1) - wf^*\|_p > \|wf(x_2) - wf^*\|_p,$$

where f^* is anti-ideal (nadir) point.

A minimax principle II5 is considered in the next section.

4. Numerical example

A bi-objective optimization problem

$$f(x) = (f_1(x^{(1)}), f_2(x^{(2)})) = \left(\sum_{j=1}^4 x_j^{(1)}, \sum_{j=1}^4 x_j^{(2)} \right) \rightarrow \min \quad (4)$$

without constraints is considered. In Table 1 the values of arguments (decision variables) x_1, x_2, x_3, x_4 are given per levels and objective functions f_1 and f_2 . We use notations $j_l^{(i)}$ instead of $x_{jl}^{(i)}$, $i=1, 2$. The superscript would not be used if it is clear to which criterion is referred.

Table 1. Subjective variables per objective functions.

Var.	Objective function f_1					Objective function f_2				
x_1	83	37	64	49		6.8	8.3	2.1	4.0	
x_2	56	73				6.7	4.6			
x_3	75	102	88			7.2	3.9	9.4		
x_4	55	65	47	80	72	1.9	3.4	2.7	3.8	4.2

All admissible solutions of MOP (4) are 120. Input data are normed per criterion to obtain dimensionless $x_{jl}^{(i)} \in [0,1]$. By $X = X^{(1)} \times X^{(2)}$, $X^{(i)} = \{x_{jl}^{(i)}\}$, $i=1, 2$, we denote the solution set. This set is shown as points in normed objective space on fig. 2.

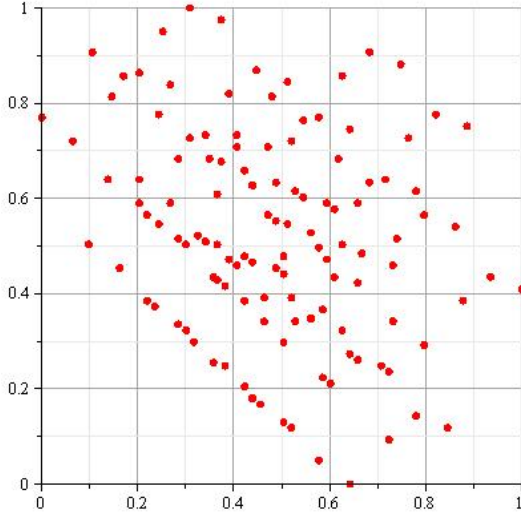


Fig. 2. Solutions of problem (4) in objective space.

The values of objective functions f_1 and f_2 are drawn as segments per solutions on fig. 3.

4.1. Compromise solutions

This problem has 18 Pareto optimal solutions that are presented on Table 2 and fig. 4. In Table 2 by $\Pi 5$ we denote minimax criterion applicable to MOP (4) as:

$$\min_{x \in X} \max_{i=1,2} \frac{f_i(x^{(i)}) - f_{i*}}{f_i^* - f_{i*}}, \quad (5)$$

where:

$x^{(i)} = (1l_1^{(i)}, 2l_2^{(i)}, 3l_3^{(i)}, 4l_4^{(i)})$, $i=1, 2$, is an argument of objective function f_i ;

$f_{1*} = f(12, 21, 31, 43) = 215$ is a minimal value of f_1 and

$f_1^* = f(11, 22, 32, 44) = 338$ is a maximal value;

$f_{2*} = f(13, 22, 32, 41) = 12.5$ is a minimal value of f_2 and

$f_2^* = f(12, 21, 33, 45) = 28.6$ is a maximal value.

To each solution a two dimensional criterion vector is corresponded, see fig. 3. According to $\Pi 5$ by (5) to every solution P the greatest value of objective functions $\max\{f_1(P), f_2(P)\}$ is

assigned, and then of all these 120 numbers we chose the minimal one.

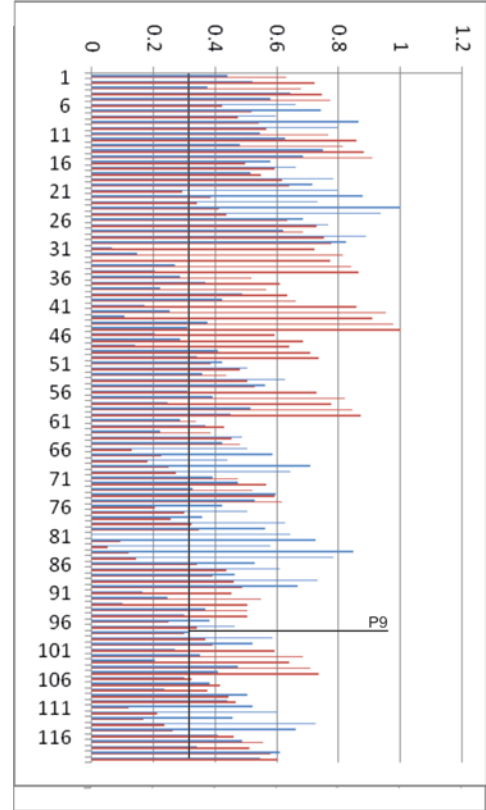


Fig. 3. Normed values of subjective functions.

For example under consideration the minimax solution is $P9 = (x^{(1)}, x^{(2)}) = (14, 21, 32, 43)$, see Table 2 and fig. 3.

Table 2. Pareto optimal solutions.

No	Solution (P)				f_1	f_2	Optimality principle
1	12	21	31	43	215	24.9	$\Pi 1, \Pi 2, \Pi 3, \Pi 4, \Pi 5$
2	12	21	31	41	223	24.1	$\Pi 5$
3	14	21	31	43	227	20.6	$\Pi 1, \Pi 3, \Pi 4, \Pi 5$
4	14	21	31	41	235	19.8	$\Pi 5$
5	13	21	31	43	242	18.7	$\Pi 3, \Pi 5$
6	14	22	31	43	244	18.5	$\Pi 1, \Pi 3, \Pi 5$
7	13	21	31	41	250	17.9	$\Pi 5$
8	14	22	31	41	252	17.7	$\Pi 5$
9	14	21	32	43	254	17.3	$\Pi 5$
10	13	22	31	43	259	16.6	$\Pi 5$
11	14	21	32	41	262	16.5	$\Pi 5$
12	13	22	31	41	267	15.8	$\Pi 5$
13	13	21	32	43	269	15.4	$\Pi 2, \Pi 5$
14	14	22	32	43	271	15.2	$\Pi 4, \Pi 5$
15	13	21	32	41	277	14.6	$\Pi 2, \Pi 4$
16	14	22	32	41	279	14.4	$\Pi 1, \Pi 5$
17	13	22	32	43	286	13.3	$\Pi 1, \Pi 3, \Pi 4, \Pi 5$
18	13	22	32	41	294	12.5	$\Pi 3, \Pi 4, \Pi 5$

All Pareto frontier is obtained as solutions of reduced SOP by one or more AOF varying weights of single objective function. The compromise solutions given in Table 2 are derived by criteria $\Pi 1 - \Pi 5$ with criteria $w_1 f_1$ and $w_2 f_2$, where $w_1=0.05, 0.10, \dots, 0.95$ and $w_2=1-w_1$. Pareto solutions derived in such a way gravitate to optimal solution of a criterion with bigger weight. Compromise solutions are numbered according to their order in Table 2 and presented on fig. 4. By example:

solution $P2 = (12, 21, 31, 41)$ is derived by minimax principle $\Pi 5$ with weights $w_1=0.2$ and $w_2=0.8$;

solution $P15 = (12, 21, 31, 43)$ is received maximizing distance to anti-ideal point (principle II4) without weights and etc.

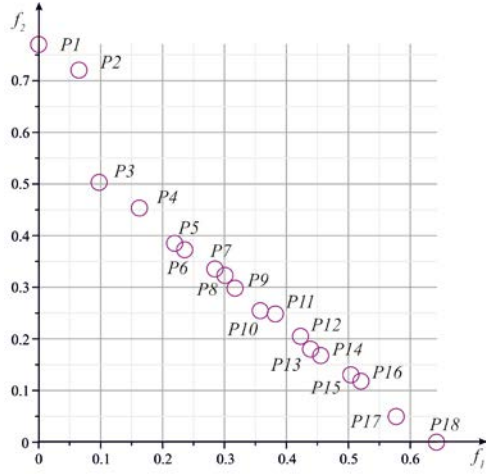


Fig. 4. Compromise solutions of problem (4).

4.2. Rankings of compromise solutions

Compromise solutions are non-comparable by their criteria values, thus we rank them by some optimality principles. Scaled (normed) objective values of Pareto solution are given in Table 3.

Table 3. Scaled objective values of Pareto optimal solutions.

Scaled criterion	Solution					
	P1	P2	P3	P4	P5	P6
f_1	0	0.065	0.098	0.163	0.219	0.236
f_2	0.770	0.720	0.503	0.453	0.385	0.373
Scaled criterion	Solution					
	P7	P8	P9	P10	P11	P12
f_1	0.285	0.301	0.317	0.358	0.382	0.423
f_2	0.335	0.323	0.298	0.255	0.248	0.205
Scaled criterion	Solution					
	P13	P14	P15	P16	P17	P18
f_1	0.439	0.455	0.504	0.520	0.577	0.642
f_2	0.180	0.168	0.130	0.118	0.050	0

Using principle II1 and second norm $\| \cdot \|_2$ with AOF

$F(f(x)) = \|f(x)\|_2$ we have:

$$\begin{aligned} \|f(P_5)\|_2 &= \sqrt{(f_1(P_5))^2 + (f_2(P_5))^2} \approx 0.606 < \|f(P_6)\|_2 \approx 0.607 < \\ \|f(P_7)\|_2 &\approx 0.621 < \|f(P_9)\|_2 \approx 0.627 < \|f(P_8)\|_2 \approx 0.629 < \\ \|f(P_4)\|_2 &\approx 0.641 < \|f(P_{10})\|_2 \approx 0.648 < \|f(P_3)\|_2 \approx 0.671 < \\ \|f(P_{11})\|_2 &\approx 0.677 < \|f(P_{12})\|_2 \approx 0.710 < \|f(P_{13})\|_2 \approx 0.629 < \\ \|f(P_{14})\|_2 &\approx 0.742 < \|f(P_{15})\|_2 \approx 0.803 < \|f(P_{16})\|_2 \approx 0.824 < \\ \|f(P_{17})\|_2 &\approx 0.901 < \|f(P_2)\|_2 \approx 0.941 < \|f(P_1)\|_2 = \|f(P_{18})\|_2 = 1, \end{aligned}$$

therefore the solutions ranking is:

$$P5 \succ P6 \succ P7 \succ P9 \succ P8 \succ P4 \succ P10 \succ P3 \succ P11 \succ P12 \succ P13 \succ P14 \succ P15 \succ P16 \succ P17 \succ P2 \succ P1 = P18.$$

The same ranking follows by II3.

Some other rankings are:

$$P3 \succ P5 \succ P4 \succ P6 \succ P7 \succ P9 \succ P10 \succ P8 \succ P13 \succ P11 \succ P12 \succ P14 \succ P15 \succ P17 \succ P16 \succ P1 = P18 \succ P2 \text{ by II1 and } \| \cdot \|_1.$$

Applying method of anti-ideal point II4 with first norm $\| \cdot \|_1$ we receive the same rating as ranking with II1 and $\| \cdot \|_2$, but by this principle with second norm we have:

$$P1 = P18 \succ P17 \succ P3 \succ P2 \succ P16 \succ P15 \succ P4 \succ P14 \succ P13 \succ P5 \succ P6 \succ P12 \succ P10 \succ P9 \succ P7 \succ P11 \succ P8.$$

We derive by minimax principle (5 without scaling, because data are normalized) a ranking:

$$P9 \succ P8 \succ P7 \succ P10 \succ P6 \succ P11 \succ P5 \succ P12 \succ P13 \succ$$

$$P4 \succ P14 \succ P3 \succ P15 \succ P16 \succ P17 \succ P18 \succ P2 \succ P1,$$

see fig. 5, where criterion values of f_1 is sketch by solid circle and f_2 - by solid diamonds.

By using scalarized AOFs it is impossible to obtain a unique ranking of Pareto solutions. The only certain information above rankings provided is that the solutions $P1$ and $P18$ seem equivalent. It is evident, that $P1$ is the solution of the first SOP

$$f_1(x^{(1)}) = \sum_{j=1}^4 x_j^{(1)} \rightarrow \min,$$

and $P18$ minimizes second criterion.

The other disturbing fact is that the values of AOFs in normed criteria are closed (less than 10^{-3}) for some solutions, which yields to approximate equivalence of such solutions, but this might be due to round off errors or errors in input data, or etc. For this reason we test some statistical techniques for classification like cluster analysis [3, 4] to join solutions in „proximity” groups (clusters) which DM would offer to the user.

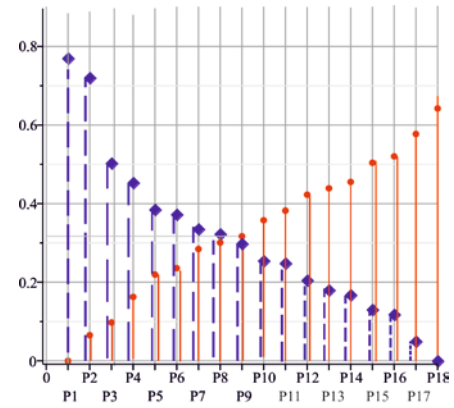


Fig. 5. Normed values of compromise solutions.

4.3. Cluster analysis

Cluster analysis (CA) is a technique for grouping dimensionless objects in clusters, where the number of clusters and their properties are usually not known. The objects are considered as points in R^n . This analysis consists of procedures and algorithms for classification, as graph algorithms, hierarchical and iterative clusterizing and others. Agglomerative hierarchical procedures start with numbers of clusters equal to the number of objects and after sequential joining finish with one cluster. A graphical representation of such procedures is a dendrogram (tree-similar diagram), see fig. 5. Analyzing dendrograms we can derive the suitable number of clusters and their elements [3, 11].

Quantitative measure of notion “similarity” (“proximity”) depends on applied metrics – Euclidian, Chebishev and Minkowski metrics and others. Similarity among objects are evaluated according their degree of proximity, i.e. by a distance (metrics) between them. As this distance smaller as the objects are closed. Commonly the calculations are based on Euclidian distance, namely the distance d between two points $x, y \in R^n$ is introduced as

$$d(x, y) = \|x - y\|_2.$$

There are different strategies for combining elements in groups and clusters, like single linkage, complete linkage and others. The distance between two clusters consisting of more than one point is considered as a distance between the two nearest points, one from each cluster. This is known as the nearest neighbor (or single linkage) method.

Considering criterion space as Euclidian plane we calculate distances d_{ij} among all compromise solutions P_i , where:

$$P_i \text{ is used as a point with coordinates } (f_1(P_i), f_2(P_i));$$

$$d_{ij} = d_{ji} = d(P_i, P_j), i, j = 1, 2, \dots, 18, j > i.$$

First we join the nearest points, i.e. the points at minimal distance. We have $d_{\min} = 0.02046 \approx d_{56} \approx d_{78} \approx d_{15,16} \approx d_{13,14}$. Hence we sequentially obtain clusters: $P5P6, P7P8, P15P16$ and $P13P14$.

Next we compute distances among new objects: $P1, \dots, P4, P5P6, P7, \dots, P18$. Distances between cluster $P5P6$ and other points are derived as follows:

$$d_{56,j} = d(P5P6, P_j) = \min\{d_{5j}, d_{6j}\} - \text{nearest neighbor.}$$

After that we merge $P7$ and $P8$, find distances between $P1, \dots, P4, P5P6, P7P8, P9, \dots, P18$ until joining $P13$ and $P14$.

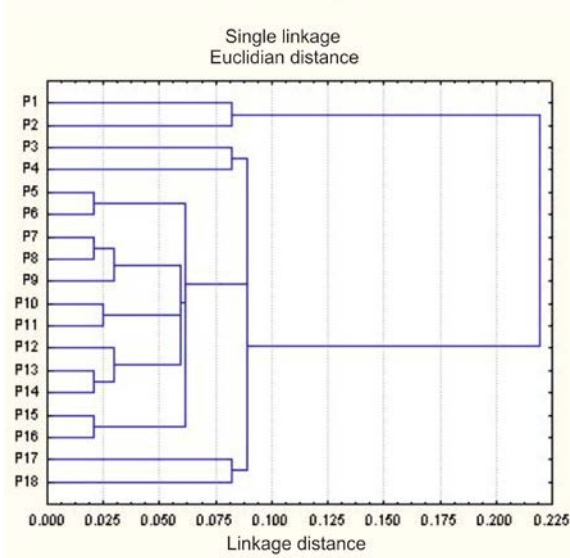


Fig. 6. Dendrogram of solutions.

Minimal distance between $P1, \dots, P4, P5P6, P7P8, P9, \dots, P12, P13P14, P15P16, P17$ and $P18$ is

$$d_{10,11} = d(P10, P11) = 0.02517.$$

Therefore groups are: $P1, \dots, P4, P5P6, P7P8, P9, P10P11, P12, P13P14, P15P16, P17$ and $P18$.

Next we join $P9$ to cluster $P10P11$ and $P12$ to $P13P14$, hence there are 11 groups: $P1, \dots, P4, P5P6, P7P8P9, P10P11P12, P13P14, P15P16, P17$ and $P18$. This clustering is shown on fig. 7 and can be determined at distance linkage > 0.03 on dendrogram, see fig. 6.

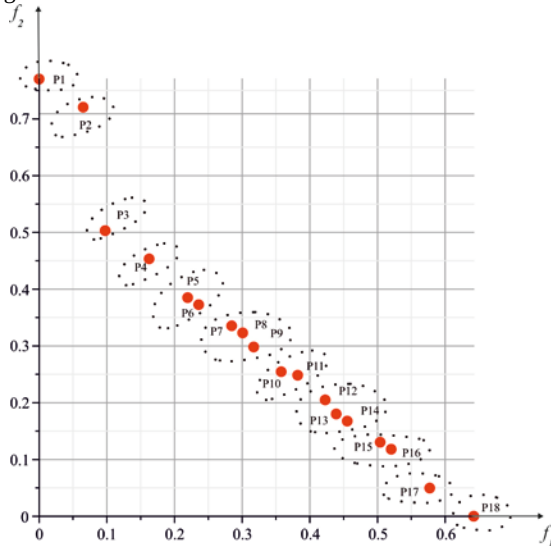


Fig. 7. A clustering of Pareto solutions.

Tracking the dendrogram we can visually examine next clusterings:

9 clusters : $P1, P2, P3, P4, P5P6, P7 - P14, P15P16, P17$ and $P18$ at linkage distance about 0.062;

7 clusters: $P1, P2, P3, P4, P5 - P16, P17$ and $P18$ at distance > 0.062 ;

4 clusters: $P1P2, P3P4, P5-P16$ and $P17P18$ at distance > 0.082 ;

2 clusters: $P1P2, P3 - P18$ at distance > 0.09 and finally one group containing all points.

It may be useful to DM to examine all possible clusterings of the compromise solutions according to the degree of proximity among the clusters and similarity of their elements, if number of clusterings is not very large.

5. Conclusion

In the report the classical principles for finding Pareto optimal solutions of MOP are systemized. DM offers users a set of compromise solutions $C(X)$, $f(C(X)) = C(Y)$, that may be ranked by same optimality criteria $\Pi1-\Pi5$ applicable for optimization of scalarized SOP. By ranking $C(Y)$ we obtain corresponding order in $C(X)$. If the user has preferences to some of criteria applying $\Pi1-\Pi5$ to weighted objective functions a weighted ranking of $C(Y)$ is performed. To support the final solution choice of MOP we suggest to use CA for clusterizing preferable solutions in "similarity" groups in scaled objective space.

References

- [1] Batishchev D.I., D.E. Shaposhnikov. Multiobjective choice with regard to individual preferences. Nizhny Novgorod, RAN IPF, 1994.
- [2] Efrani, T., Utyuzhnikov, S.V., Directed search domain: a method for even generation of the Pareto frontier in multiobjective optimization. Engineering Optimization, Vol. 43, No. 5, 2011, 467-484.
- [3] Härdle, W., Simar, L., Applied multivariate statistical analysis, MD&TECH, 2003.
- [4] Kalashnikov, A.E. Dialogic system for multiobjective optimization of technological processes. Dissertation for doctoral degree in technical science. MGIS, Moscow, 2004.
- [5] Malakov, I. Methodology for choice of optimal assembly structural variant. Research work qualifying to receive the title of Professor, Sofia, 2009.
- [6] Marichal, J.-L. Aggregation operators for multicriteria decision aid. Dissertation for doctoral degree in sciences, Universite de Liege, 1999.
- [7] Mashunin, Yu.K. Vector optimization methods and models. Moscow, Nauka, 1986.
- [8] Mihalevich, V.S., V.L. Volkovich. Computational methods for research and design of complex systems. Moscow, Nauka, 1982.
- [9] Nogin, V.D. Decision making in multiobjective environment: quantitative approach. Moscow, FIZMATLIT, 2002.
- [10] Stoyanov, S.K. Technological process optimization. Sofia, Technika, 1993.
- [11] Tryfos, P. Cluster analysis. Methods for business analysis and forecasting: York University, 1998.
- [12] Zajchenko, Yu.P. Operations research: Fuzzy optimizations. Kiev, Visha skola, 1991.